



TITLE:

ON THE EXTENSIONS OF  
 $\mathcal{W}_{n,A}$  BY  
 $\mathbb{G}_{m,A}$  (Rigid  
Geometry and Group Action)

AUTHOR(S):

Sekiguchi, Tsutomu; Suwa, Noriyuki

---

CITATION:

Sekiguchi, Tsutomu ...[et al]. ON THE EXTENSIONS OF  $\mathcal{W}_{n,A}$  BY  $\mathbb{G}_{m,A}$  (Rigid Geometry and Group Action). 数理解析研究所講究録 1998, 1073: 84-97

ISSUE DATE:

1998-12

URL:

<http://hdl.handle.net/2433/62598>

RIGHT:

# ON THE EXTENSIONS OF $\mathcal{W}_{n,A}$ BY $\mathbb{G}_{m,A}$

TSUTOMU SEKIGUCHI\*) AND NORIYUKI SUWA†)

## 1. MOTIVATION

Let  $(A, \mathfrak{M})$  be a DVR with  $K = Q(A)$  of characteristic 0 and  $k = A/\mathfrak{M}$  of characteristic  $p > 0$ .

The so called Artin-Schreier-Witt exact sequence

$$(1) \quad 0 \rightarrow \mathbb{Z}/p^n \rightarrow W_{n,k} \xrightarrow{F^n - \text{id}} W_{n,k} \rightarrow 0$$

describes any étale  $p^n$ -cyclic coverings, where  $W_{n,k}$  is the group scheme over  $k$  of Witt vectors of length  $n$  and  $F$  is the Frobenius endomorphism.

On the other hand, when  $K$  contains  $\mu_{p^n}$ , any étale  $p^n$ -cyclic coverings are described by the Kummer sequence

$$(2) \quad 0 \rightarrow \mu_{p^n, K} \rightarrow \mathbb{G}_{m, K} \xrightarrow{\theta_{p^n}} \mathbb{G}_{m, K} \rightarrow 0.$$

But we do not like to have two Gods (1) and (2) in the world.

In fact, we can construct a Kummer-Artin-Schreier-Witt exact sequence over DVR  $A = \mathbb{Z}_{(p)}[\mu_{p^n}]$ :

$$(3) \quad 0 \rightarrow (\mathbb{Z}/p^n)_A \rightarrow \mathcal{W}_n \rightarrow \mathcal{W}_n/(\mathbb{Z}/p^n)_A \rightarrow 0$$

with an exact sequence of Kummer type as the generic fibre:

$$(4) \quad 0 \rightarrow \mu_{p^n, K} \rightarrow (\mathbb{G}_{m, K})^n \rightarrow (\mathbb{G}_{m, K})^n \rightarrow 0$$

and with (1) as the special fibre (cf. [12, 15]).

In  $n = 1$  case, the exact sequence (3) is given explicitly as follows:

Let  $\zeta$  be a primitive  $p$ -th root of unity,  $\lambda = \zeta - 1$  and  $A = \mathbb{Z}_{(p)}[\zeta]$ . We define  $\mathcal{W}_1$  by the group scheme

$$\mathcal{G}^{(\lambda)} = \text{Spec} A[x, \frac{1}{\lambda x + 1}]$$

with group law  $x \cdot y = x + y + \lambda xy$ . Then (3) is given by

$$(5) \quad 0 \rightarrow (\mathbb{Z}/p)_A \rightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\Psi} \mathcal{G}^{(\lambda)}/(\mathbb{Z}/p) \cong \mathcal{G}(\lambda^p) \rightarrow 0,$$

\*) Partially supported by Grant-in-Aid for Scientific Research #08640059

†) Partially supported by Grant-in-Aid for Scientific Research #09640066

where  $\Phi(x) = \frac{1}{\lambda^p} \{(\lambda x + 1)^p - 1\}$ . The exact sequence (3) for general  $n$  is given by taking suitable extensions step by step starting from the exact sequence (5); that is to say, if we construct  $\mathcal{W}_n$  for an  $n$ , then the next  $\mathcal{W}_{n+1}$  is given by an extension of  $\mathcal{W}_n$  by  $\mathcal{G}^{(\lambda)}$ :

$$(6) \quad 0 \rightarrow \mathcal{G}^{(\lambda)} \rightarrow \mathcal{W}_{n+1} \rightarrow \mathcal{W}_n \rightarrow 0 \in \text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}).$$

On the other hand, some matters concerning of  $\mathcal{G}^{(\lambda)}$  can be calculated by using the exact sequence

$$(7) \quad 0 \rightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\alpha^{(\lambda)}} \mathbb{G}_{m,A} \xrightarrow{r^{(\lambda)}} i_* \mathbb{G}_{A/\lambda} \rightarrow 0,$$

where  $i : \text{Spec} A/\lambda \hookrightarrow \text{Spec} A$  is the canonical inclusion,  $\alpha^{(\lambda)}(x) = \lambda x + 1$  and  $r^{(\lambda)}t \equiv t \pmod{\lambda}$ .

In fact, using this exact sequence (7), we can obtain a long exact sequence

$$(8) \quad \begin{aligned} 0 &\rightarrow \text{Hom}(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) \rightarrow \text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A}) \rightarrow \text{Hom}(\mathcal{W}_n, i_* \mathbb{G}_{m,A/\lambda}) \\ &\xrightarrow{\partial} \text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) \rightarrow \text{Ext}^1(\mathcal{W}_n, \mathbb{G}_{m,A}). \end{aligned}$$

Here we have  $\text{Ext}^1(\mathcal{W}_n, \mathbb{G}_{m,A}) = 0$  by Hilbert theorem 90. Therefore for our purpose to search  $\mathcal{W}_{n+1}$ , to calculate

$$\text{Hom}(\mathcal{W}_n, i_* \mathbb{G}_{m,A/\lambda}) \cong \text{Hom}(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$$

is important.

Moreover to determine explicitly the quotient  $\mathcal{W}_n/(\mathbb{Z}/p^n)_A$  is crucial when we apply our theory to the lifting problems of  $p^n$ -cyclic coverings of curves as was expanded by B. Green and M. Matignon [4]. When once we construct the quotient  $\mathcal{W}_n/(\mathbb{Z}/p^n)_A$ , the next one  $\mathcal{W}_{n+1}/(\mathbb{Z}/p^{n+1})_A$  is given in  $\text{Ext}^1(\mathcal{W}_n/(\mathbb{Z}/p^n)_A, \mathcal{G}^{(\lambda^p)})$ , and it is fixed explicitly by calculating  $\text{Hom}(\mathcal{W}_{n,A/\lambda^p}, \mathbb{G}_{m,A/\lambda^p})$ .

Our aim of this report is to determine explicitly the groups  $\text{Hom}(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$  and  $\text{Ext}^1(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$ .

## 2. THE STRUCTURE OF $\mathcal{W}_n$

By using the exact sequence (8), for  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathfrak{M} \setminus \{0\}$ ,  $\mathcal{W}_n$  can be written in the form:

$$(9) \quad \mathcal{W}_n = \text{Spec} A[X_0, X_1, \dots, X_{n-1}, \frac{1}{\lambda_1 X_0 + 1}, \frac{1}{\lambda_2 X_1 + D_1(X_0)}, \frac{1}{\lambda_3 X_2 + D_2(X_0, X_1)}, \dots, \frac{1}{\lambda_n X_{n-1} + D_{n-1}(X_0, \dots, X_{n-2})}]$$

where for each  $i = 1, 2, \dots, n-1$ ,

$$D_i : \text{Spec} A / \lambda_{i+1} [X_0, \dots, X_{i-1}, \frac{1}{\lambda_1 X_0 + 1}, \dots, \frac{1}{\lambda_i X_{i-1} + D_{i-1}}] \rightarrow \mathbb{G}_{m,A/\lambda_{i+1}}$$

is a homomorphism and we understand that  $D_0 = 1$ . The group law of  $\mathcal{W}_n$  is that which makes the map

$$\begin{aligned} \mathcal{W}_n &\rightarrow (\mathbb{G}_{m,A})^n \\ \mathbf{x} = (x_0, \dots, x_{n-1}) &\mapsto (\lambda_1 x_1 + 1, \lambda_2 x_1 + D_1(\mathbf{x}), \dots, \lambda_n x_{n-1} + D_{n-1}(\mathbf{x})) \end{aligned}$$

a homomorphism. One can refer to [15] for the details.

### 3. DEFORMED ARTIN-HASSE EXPONENTIAL SERIES

Let  $W_n$  (resp.  $\widehat{W}_n$ ) be the group scheme (resp. the formal group scheme) over  $\mathbb{Z}$  of Witt vectors of length  $n$ , and  $W$  (resp.  $\widehat{W}$ ) the group scheme (resp. the formal group scheme) of Witt vectors over  $\mathbb{Z}$ , and let  $\mathbb{G}_m$  (resp.  $\widehat{\mathbb{G}}_m$ ) be the multiplicative group scheme (resp. the multiplicative formal group scheme) over  $\mathbb{Z}$ .

We denote the Witt polynomials by

$$\begin{aligned} \Phi_0(\mathbf{X}) &= X_0 \\ \Phi_1(\mathbf{X}) &= X_0^p + pX_1 \\ &\vdots \\ \Phi_n(\mathbf{X}) &= X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^n X_n \\ &\vdots \end{aligned}$$

Let  $F$  be the Frobenius endomorphism defined by

$$\begin{array}{ccccc} F := \Phi^{-1} \circ \Phi^{(1)} : & W & \xrightarrow{\Phi^{(1)}} & \mathbb{G}_a^\infty & \xleftarrow{\Phi} W \\ & \mathbf{x} & \mapsto & (\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x}), \dots) & \\ & & & (\Phi_0(\mathbf{y}), \Phi_1(\mathbf{y}), \dots) & \longleftarrow \mathbf{y} \end{array}$$

We note that  $F$  is also an endomorphism of  $\widehat{W}$ .

For later use, we define a morphism  $[p] : W \rightarrow W$  by

$$[p]\mathbf{b} := (0, b_0^p, b_1^p, \dots)$$

for a vector  $\mathbf{b} = (b_0, b_1, \dots) \in W(A)$ . Note that if  $A$  is an  $\mathbb{F}_p$ -algebra,  $[p]\mathbf{b}$  is nothing but  $p\mathbf{b}$ . Moreover, for a vector  $\mathbf{a} = (a_0, a_1, \dots) \in W(A)$ , we define a map  $T_{\mathbf{a}} : W(A) \rightarrow W(A)$  by

$$\Phi_n(T_{\mathbf{a}}\mathbf{b}) = a_0^{p^n} \Phi_n(\mathbf{b}) + pa_1^{p^{n-1}} \Phi_{n-1}(\mathbf{b}) + \dots + p^n a_n \Phi_0(\mathbf{b})$$

for  $\mathbf{b} \in W(A)$ . Then we can easily see the following.

**Lemma 3.1.** *Actually,  $T_{\mathbf{a}} : W(A) \rightarrow W(A)$  is a well-defined endomorphism.*

Let  $\bar{\Lambda}$  denote the Witt vector  $(\Lambda, 0, 0, \dots)$  with coefficients in  $\mathbb{Z}[\Lambda]$  and  $F^{(\Lambda)}$  the endomorphism  $F - \bar{\Lambda}^{p-1}$  of the group scheme  $W_{\mathbb{Z}[\Lambda]}$ .

The so called Artin-Hasse exponential series is given by

$$\begin{aligned} E_p(X) &:= \exp \left( X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \dots \right) \\ &= e^X e^{\frac{X^p}{p}} e^{\frac{X^{p^2}}{p^2}} \dots \in \mathbb{Z}_{(p)}[[X]]. \end{aligned}$$

Now we define a formal power series  $E_p(U, \Lambda; X)$  in  $\mathbb{Q}[U, \Lambda][[X]]$  by

$$E_p(U, \Lambda; X) := (1 + \Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} \left( 1 + \Lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k} \left( \left( \frac{U}{\Lambda} \right)^{p^k} - \left( \frac{U}{\Lambda} \right)^{p^{k-1}} \right)}.$$

In our argument, one of the crucial points is to decide the integrality of this kind of series. For checking the integrality, Hazewinkel's lemma (cf. [2, §2]) is almost almighty in our case.

**Lemma 3.2** ([2, (2.3.3)]). *Let  $A$  be an integral domain containing  $\mathbb{Z}_{(p)}$ , and  $\sigma : K = Q(A) \rightarrow K = Q(A)$  be a  $\mathbb{Z}_{(p)}$ -algebra homomorphism such that  $\sigma(f) \equiv f \pmod{pA}$  for any  $f \in A$ . Let  $d(X) = d_0 X + d_1 X^{p^1} + \dots \in A[\frac{1}{p}][[X]]$ . Then*

$$\exp(d(X)) = 1 + d(X) + \frac{1}{2!} d(X)^2 + \dots \in A[[X]]$$

*if and only if there exist  $b_i \in A$  ( $i = 0, 1, \dots$ ) such that  $d_0 = b_0$ , and  $d_n = b_n + \frac{1}{p} \sigma(d_{n-1}) \in A$  for  $n \geq 1$ .*

By using this lemma, we can see that  $E_p(U, \Lambda; X) \in \mathbb{Z}_{(p)}[U, \Lambda][[X]]$ . Easily we can see that  $E_p(1, 0; X) = E_p(X)$ , that is to say,  $E_p(U, \Lambda; X)$  gives a deformation of the Artin-Hasse exponential series  $E_p(X)$ .

Let  $A$  be a  $\mathbb{Z}_{(p)}$ -algebra,  $\lambda \in A$  and  $\mathbf{a} = (a_0, a_1, \dots) \in W(A)$ . We define a formal power series  $E_p(\mathbf{a}, \lambda; X)$  in  $A[[X]]$  by

$$\begin{aligned} (10) \quad E_p(\mathbf{a}, \lambda; X) &:= \prod_{k=0}^{\infty} E_p(a_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{\frac{a_0}{\lambda}} \prod_{k=1}^{\infty} \left( 1 + \lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} F^{(\lambda)} \mathbf{a}} \end{aligned}$$

Then the boundary of this power series  $E_p(\mathbf{a}, \lambda; X)$  is given by the following.

(11)

$$\begin{aligned} (\partial E_p(\mathbf{a}, \lambda; \cdot))(X, Y) &= \frac{E_p(\mathbf{a}, \lambda; X)E_p(\mathbf{a}, \lambda; Y)}{E_p(\mathbf{a}, \lambda; X + Y + \lambda XY)} \\ &= \prod_{k=1}^{\infty} \left( \frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} F^{(\lambda)} \mathbf{a}} \end{aligned}$$

Now replacing  $F^{(\lambda)} \mathbf{a}$  with a Witt vector  $\mathbf{b} = (b_0, b_1, \dots)$  in the right hand side of (11), we define a cocycle as follows.

$$(12) \quad F_p(\mathbf{b}, \lambda; X, Y) := \prod_{k=1}^{\infty} \left( \frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} \mathbf{b}}$$

Again using the integrality lemma, we can see that

$$F_p(\mathbf{b}, \lambda; X, Y) \in \mathbb{Z}_{(p)}[\mathbf{b}, \lambda][[X, Y]].$$

#### 4. DETERMINATION OF $\text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A})$ AND $H_0^2(\mathcal{W}_n, \mathbb{G}_{m,A})$

Let  $A$  be a  $\mathbb{Z}_{(p)}$ -algebra and  $\lambda \in A$ . By (10) and (11), we can define homomorphisms

$$\xi_0^1 : \text{Ker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) \rightarrow \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; X)$$

and, when  $\lambda$  is nilpotent,

$$\xi_0^1 : \text{Ker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) \rightarrow \text{Hom}_{A\text{-gr}}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}); \mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; X).$$

Moreover, by (12), we can define homomorphisms

$$\xi_1^1 : \text{Coker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) \rightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y)$$

and, when  $\lambda$  is nilpotent,

$$\xi_1^1 : \text{Coker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) \rightarrow H_0^2(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}); \mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y).$$

Under these notations, we gave the result in the one-dimensional case in the previous paper [16] as in the following style.

**Theorem 4.1.** *Let  $A$  be a  $\mathbb{Z}_{(p)}$ -algebra and  $\lambda \in A$ . Then the homomorphisms*

$$\begin{aligned} \xi_0^1 : \text{Ker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) &\rightarrow \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}), \\ \xi_1^1 : \text{Coker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) &\rightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \end{aligned}$$

are bijective. Moreover, if  $\lambda$  is nilpotent, the homomorphisms

$$\begin{aligned}\xi_0^1 : \text{Ker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) &\rightarrow \text{Hom}_{A\text{-gr}}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}), \\ \xi_1^1 : \text{Coker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) &\rightarrow H_0^2(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A})\end{aligned}$$

are bijective.

For general  $n$ , we can consider the both of  $\text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A})$ ,  $H_0^2(\mathcal{W}_n, \mathbb{G}_{m,A})$  and  $\text{Hom}(\widehat{\mathcal{W}}_n, \widehat{\mathbb{G}}_{m,A})$ ,  $H_0^2(\widehat{\mathcal{W}}_n, \widehat{\mathbb{G}}_{m,A})$ , but for simplicity hereafter we treat the first them only.

Next we look at  $n = 2$  case. Let  $\lambda_1, \lambda_2 \in A$ , and assume that  $\lambda_1$  is nilpotent in  $A/\lambda_2$ . By (9) and Theorem 4.1, an extension

$$(13) \quad 0 \rightarrow \mathcal{G}^{(\lambda_2)} \rightarrow \mathcal{W}_2 \rightarrow \mathcal{G}^{(\lambda_1)} \rightarrow 0 \in \text{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)})$$

is given by

$$(14) \quad \mathcal{W}_2 = \text{Spec}A[X, Y, \frac{1}{\lambda_1 X + 1}, \frac{1}{\lambda_2 Y + D(X)}]$$

where  $D(X) = E_p(\mathbf{a}, \lambda_1; X)$  and  $\mathbf{a} \in \widehat{W}(A/\lambda_2)$  with  $F^{(\lambda_1)}\mathbf{a} = 0 \in \widehat{W}(A/\lambda_2)$ . Now we put  $F^{(\lambda_1)}\mathbf{a} = (\lambda_2 a'_0, \lambda_2 a'_1, \dots)$  and  $\mathbf{a}' = (a'_0, a'_1, \dots)$ . We define an endomorphism  $U_2 = U_2(\lambda_1, \lambda_2; \mathbf{a}') : \widehat{W}(A)^2 \rightarrow \widehat{W}(A)^2$  by

$$U_2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)} & -T\mathbf{a}' \\ 0 & F^{(\lambda_2)} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)}\alpha - T\mathbf{a}'\beta \\ F^{(\lambda_2)}\beta \end{pmatrix}.$$

For  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \widehat{W}(A)^2$ , we define a formal power series  $E_p(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y)$  by

$$E_p(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y) := E_p(\alpha, \lambda_1; X) E_p(\beta, \lambda_2; \frac{Y}{D(X)}).$$

When we put  $(s, t) := (x_1, y_1) + (x_2, y_2) \in \mathcal{W}_2$ , we can easily see that

$$\begin{aligned}s &= x_1 + x_2 + \lambda_1 x_1 x_2, \\ \frac{t}{D(s)} &= \frac{y_1}{D(x_1)} + \frac{y_2}{D(x_2)} + H_1(x_1, x_2) \in \mathcal{G}^{(\lambda_2)},\end{aligned}$$

where

$$\begin{aligned}H_1(x_1, x_2) &= \frac{1}{\lambda_2} \left\{ \frac{D(x_1)D(x_2)}{D(x_1 + x_2 + \lambda_1 x_1 x_2)} - 1 \right\} \\ &= \frac{1}{\lambda_2} \left\{ F_p(F^{(\lambda_1)}\mathbf{a}, \lambda_1; (x_1, x_2)) - 1 \right\}.\end{aligned}$$

Moreover, for  $F = F_p(\mathbf{b}, \lambda_1; X_1, X_2)$ , we define

$$[p]F := F_p([p]\mathbf{b}, \lambda_1; X_1, X_2),$$

and

$$G_p(\delta, \lambda_2; F) := \prod_{k=1}^{\infty} \left( \frac{1 + (F-1)^{p^k}}{[p]^k F} \right)^{\frac{1}{p^k \lambda_2^{p^k}} \Phi_{k-1} \delta}.$$

Then using again the integrality lemma, we can see that

$$G_p(\delta, \lambda_2; F) \in \mathbb{Z}_{(p)}[\mathbf{a}', \lambda_1, \lambda_2, \delta][[X_1, X_2]].$$

Under these notations, we can show that the boundary of  $E_p\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y\right)$  is given by

(15)

$$\begin{aligned} & (\partial E_p\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; \cdot, \cdot\right))(X_1, Y_1, X_2, Y_2) \\ &= \frac{E_p\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X_1, Y_1\right) E_p\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X_2, Y_2\right)}{E_p\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; (X_1, Y_1) + (X_2, Y_2)\right)} \\ &= F_p(F^{(\lambda_1)} \alpha - T_{\mathbf{a}'} \beta, \lambda_1; (X_1, X_2)) F_p(F^{(\lambda_2)} \beta, \lambda_2; \left(\frac{Y_1}{D(X_1)}, \frac{Y_2}{D(X_2)}\right)) \times \\ & \quad F_p(F^{(\lambda_2)} \beta, \lambda_2; \left(H_1, \frac{Y_1}{D(X_1)} + \frac{Y_2}{D(X_2)}\right)) G_p(-F^{(\lambda_2)} \beta, \lambda_2; F). \end{aligned}$$

Now arranging the equality (15), we define a cocycle by

$$\begin{aligned} (16) \quad & F_p\left(\begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; (X_1, Y_1), (X_2, Y_2)\right) \\ &= F_p(\gamma, \lambda_1; (X_1, X_2)) F_p(\delta, \lambda_2; \left(\frac{Y_1}{D(X_1)}, \frac{Y_2}{D(X_2)}\right)) \times \\ & \quad F_p(\delta, \lambda_2; \left(H_1, \frac{Y_1}{D(X_1)} + \frac{Y_2}{D(X_2)}\right)) G_p(-\delta, \lambda_2; F). \end{aligned}$$

By the equality (15), we can define the homomorphisms

$$\begin{aligned} \xi_0^2 : \text{Ker} \left( \widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) &\rightarrow \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto E_p\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y\right) \end{aligned}$$



and

$$\begin{aligned} \xi_1^2 : \text{Coker} \left( \widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) &\rightarrow H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} &\mapsto F_p \left( \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; (X_1, Y_1), (X_2, Y_2) \right). \end{aligned}$$

Then we can obtain a commutative diagram:

$$\begin{array}{ccccccc} (17) & 0 & \longrightarrow & \text{Ker} F^{(\lambda_1)} & \longrightarrow & \text{Ker} U_2 & \longrightarrow & \text{Ker} F^{(\lambda_2)} & \longrightarrow \\ & & & \xi_0^1 \downarrow & & \xi_0^2 \downarrow & & \xi_0^1 \downarrow & \\ 0 & \longrightarrow & \text{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{G}^{(\lambda_2)}, \mathbb{G}_{m,A}) & \xrightarrow{\partial} \\ & & & \longrightarrow & \text{Coker} F^{(\lambda_2)} & \longrightarrow & \text{Coker} U_2 & \longrightarrow & \text{Coker} F^{(\lambda_2)} & \longrightarrow & 0 \\ & & & \xi_1^1 \downarrow & & \xi_1^2 \downarrow & & \xi_1^1 \downarrow & \\ & \xrightarrow{\partial} & H_0^2(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{G}^{(\lambda_2)}, \mathbb{G}_{m,A}), \end{array}$$

where the second horizontal line is the exact sequence deduced from (13), and the first horizontal line is the exact sequence defined by the following maps in order:

$$\begin{aligned} \alpha &\mapsto \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, & \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \beta, & \beta &\mapsto T_{a'} \beta, \\ \gamma &\mapsto \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, & \begin{pmatrix} \gamma \\ \delta \end{pmatrix} &\mapsto \delta. \end{aligned}$$

By the commutative diagram (17) and Theorem 4.1, we can show the following.

**Theorem 4.2.** *The homomorphisms*

$$\xi_0^2 : \text{Ker} \left( \widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) \rightarrow \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A})$$

and

$$\xi_1^2 : \text{Coker} \left( \widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) \rightarrow H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A})$$

are bijective.

Before we give the final form of the theorem, we will explain the situation by looking at the  $n = 3$  case.

By (9), an extension

$$(18) \quad 0 \rightarrow \mathcal{G}^{(\lambda_3)} \rightarrow \mathcal{W}_3 \rightarrow \mathcal{W}_2 \rightarrow 0$$

is given by

$$\mathcal{W}_2 = \text{Spec}A[X, Y, \frac{1}{1 + \lambda_1 X}, \frac{1}{D_1(X) + \lambda_2 Y}]$$

and

$$\mathcal{W}_3 = \text{Spec}A[X, Y, Z, \frac{1}{1 + \lambda_1 X}, \frac{1}{D_1(X) + \lambda_2 Y}, \frac{1}{D_2(X, Y) + \lambda_3 Z}].$$

Moreover, by Theorems 4.1, 4.2,  $D_1$  and  $D_2$  are given by

$$\begin{aligned} D_1(X) &= E_p(\mathbf{a}, \lambda_1; X), \\ D_2(X, Y) &= E_p\left(\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y\right), \end{aligned}$$

with

$$\begin{aligned} F^{(\lambda_1)}\mathbf{a} &= \mathbf{0} \in \widehat{W}(A/\lambda_2), \\ U_2\left(\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}\right) &= \begin{pmatrix} F^{(\lambda_1)} & -T\mathbf{a}' \\ 0 & F^{(\lambda_2)} \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \mathbf{0} \in \widehat{W}(A/\lambda_3), \end{aligned}$$

where  $\mathbf{a}'$  is defined by  $(\lambda_2 a'_0, \lambda_2 a'_2, \dots) = F^{(\lambda_1)}\mathbf{a}$ . Now we define  $\mathbf{b}'$  and  $\mathbf{c}'$  by  $(\lambda_3 b'_0, \lambda_3 b'_2, \dots) = F^{(\lambda_2)}\mathbf{b} - T\mathbf{a}'\mathbf{c}$  and  $(\lambda_3 c'_0, \lambda_3 c'_1, \dots) = F^{(\lambda_3)}\mathbf{c}$ .

Again we define a new power series by

$$E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X, Y, Z\right) = E_p(\boldsymbol{\alpha}, \lambda_1; X)E_p(\boldsymbol{\beta}, \lambda_2; \frac{Y}{D_1(X)})E_p(\boldsymbol{\gamma}, \lambda_3; \frac{Z}{D_2(X, Y)}).$$

Then the boundary of this series can be calculated as follows.

$$\begin{aligned} (19) \quad & \left( \partial E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; \cdot, \cdot, \cdot\right) (X_1, Y_1, Z_1, X_2, Y_2, Z_2) \right. \\ & \quad \left. = \frac{E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1\right)E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_2, Y_2, Z_2\right)}{E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; (X_1, Y_1, Z_1) + (X_2, Y_2, Z_2)\right)} \right) \end{aligned}$$

$$\begin{aligned}
&= F_p(F^{(\lambda_1)}\alpha, \lambda_1; X_1, X_2)F_p(F^{(\lambda_2)}\beta, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times \\
&\quad F_p(F^{(\lambda_2)}\beta, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)})E_p(\beta, \lambda_2; H_1)^{-1} \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)}))E_p(\gamma, \lambda_3; H_2)^{-1} \\
&= F_p(F^{(\lambda_1)}\alpha - T_{\mathbf{a}'}\beta - T_{\mathbf{b}'}\gamma, \lambda_1; X_1, X_2) \times \\
&\quad F_p(F^{(\lambda_2)}\beta - T_{\mathbf{b}'}\beta, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times \\
&\quad F_p(F^{(\lambda_2)}\beta - T_{\mathbf{b}'}\gamma, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)})) \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) \times \\
&\quad G_p(F^{(\lambda_2)}\beta - T_{\mathbf{b}'}\gamma, \lambda_2; F_p(F^{(\lambda_1)}\alpha, \lambda_1; X_1, X_2))^{-1} \times \\
&\quad G_p(F^{(\lambda_3)}\gamma, \lambda_3; \lambda_3 H_2 + 1)^{-1}.
\end{aligned}$$

Again arranging the equation (19), we define a cocycle by

$$\begin{aligned}
&F_p\left(\begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1, X_2, Y_2, Z_2\right) \\
&= F_p(\delta, \lambda_1; X_1, X_2)F_p(\epsilon, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times \\
&\quad F_p(\epsilon, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)}))F_p(\zeta, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times \\
&\quad F_p(\zeta, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) \times \\
&\quad G_p(\epsilon, \lambda_2; F_p(F^{(\lambda_1)}\alpha, \lambda_1; X_1, X_2))^{-1}G_p(\zeta, \lambda_3; \lambda_3 H_2 + 1)^{-1}.
\end{aligned}$$

Now we define an endomorphism  $U_3 = U(\lambda_1, \lambda_2, \lambda_3, \mathbf{a}', \mathbf{b}', \mathbf{c}') : \widehat{W}(A)^3 \rightarrow \widehat{W}(A)^3$

by

$$U_3 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)} & -T_{\mathbf{a}'} & -T_{\mathbf{b}'} \\ 0 & F^{(\lambda_1)} & -T_{\mathbf{c}'} \\ 0 & 0 & F^{(\lambda_3)} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)}\alpha - T_{\mathbf{a}'}\beta - T_{\mathbf{b}'}\gamma \\ F^{(\lambda_1)}\beta - T_{\mathbf{c}'}\gamma \\ F^{(\lambda_3)}\gamma \end{pmatrix}.$$

Then by (19), we can define homomorphisms

$$\begin{aligned} \xi_0^3 : \text{Ker}(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3) &\rightarrow \text{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &\mapsto E_p\left(\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X, Y, Z\right), \\ \xi_1^3 : \text{Coker}(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3) &\rightarrow H_0^2(\mathcal{W}_3, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix} &\mapsto F_p\left(\begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1, X_2, Y_2, Z_2\right). \end{aligned}$$

Then similarly as in the case of  $n = 2$ , we have a commutative diagram:

$$\begin{array}{ccccccc} (20) & 0 & \longrightarrow & \text{Ker}U_2 & \longrightarrow & \text{Ker}U_3 & \longrightarrow & \text{Ker}F^{(\lambda_3)} & \longrightarrow \\ & & & \xi_0^2 \downarrow & & \xi_0^3 \downarrow & & \xi_0^1 \downarrow & \\ 0 & \longrightarrow & \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{G}^{(\lambda_3)}, \mathbb{G}_{m,A}) & \xrightarrow{\partial} \\ & & & & & & & & \\ & \longrightarrow & \text{Coker}U_2 & \longrightarrow & \text{Coker}U_3 & \longrightarrow & \text{Coker}F^{(\lambda_3)} & \longrightarrow & 0 \\ & & \xi_1^2 \downarrow & & \xi_1^3 \downarrow & & \xi_1^1 \downarrow & & \\ & \xrightarrow{\partial} & H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{W}_3, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{G}^{(\lambda_3)}, \mathbb{G}_{m,A}), & & \end{array}$$

with the second exact horizontal line obtained by the exact sequence (18) and the first horizontal line defined in order as follows:

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, & \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &\mapsto \gamma, & \gamma &\mapsto \begin{pmatrix} T_{\mathbf{b}'}\gamma \\ T_{\mathbf{c}'}\gamma \end{pmatrix}, \\ \begin{pmatrix} \delta \\ \epsilon \end{pmatrix} &\mapsto \begin{pmatrix} \delta \\ \epsilon \\ 0 \end{pmatrix}, & \begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix} &\mapsto \zeta. \end{aligned}$$

By the commutative diagram (20) and Theorem 4.2, we have the result in the three-dimensional case.

**Theorem 4.3.** *The homomorphisms*

$$\xi_0^3 : \text{Ker} \left( \widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3 \right) \rightarrow \text{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A})$$

and

$$\xi_1^3 : \text{Coker} \left( \widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3 \right) \rightarrow H_0^2(\mathcal{W}_3, \mathbb{G}_{m,A})$$

are bijective.

Then as one can guess the general result, we have the final form as follows.

Let  $\mathcal{W}_n$  is a group scheme over  $A$  obtained successively by

$$(21) \quad 0 \rightarrow \mathcal{G}^{(\lambda_{i+1})} \rightarrow \mathcal{W}_{i+1} \rightarrow \mathcal{W}_i \rightarrow 0,$$

for  $i = 1, 2, \dots, n-1$ , where  $\mathcal{W}_1 = \mathcal{G}^{(\lambda_1)}$ . Then by (9), each  $\mathcal{W}_i$  ( $i = 1, 2, \dots, n$ ) is given by

$$\mathcal{W}_i = \text{Spec} A[X_0, X_1, \dots, X_{i-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{i-1}(X_0, \dots, X_{i-2}) + \lambda_i X_{i-1}}],$$

where  $D_k(X_0, \dots, X_{k-1}) : \mathcal{W}_{k,A/\lambda_{k+1}} \rightarrow \mathbb{G}_{m,A/\lambda_{k+1}}$  is a homomorphism for  $k = 1, \dots, i-1$ . Here we understand that  $D_0 = 1$ . Now we assume that for  $1 \leq i \leq n-1$ , each  $D_i(X_0, X_1, \dots, X_{i-1})$  is given by

$$\begin{aligned} D_i(X_0, X_1, \dots, X_{i-1}) &= E_p \left( \begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_i^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1} \right) \\ &:= \prod_{\ell=1}^i E_p(a_\ell^i, \lambda_\ell; \frac{X_{\ell-1}}{D_{\ell-1}(X_0, \dots, X_{\ell-2})}), \end{aligned}$$

and

$$\begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_i^i \end{pmatrix} \in \text{Ker} \left( U_i := \begin{pmatrix} F^{(\lambda_1)} & -T_{a_1^1} & -T_{a_1^2} & \cdots & -T_{a_1^{i-1}} \\ 0 & F^{(\lambda_2)} & -T_{a_2^2} & \cdots & -T_{a_2^{i-1}} \\ 0 & 0 & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & -T_{a_{i-1}^{i-1}} \\ bm0 & 0 & 0 & \ddots & F^{(\lambda_{i-1})} \end{pmatrix} : \widehat{W}(A/\lambda_i)^i \rightarrow \widehat{W}(A/\lambda_i)^i \right).$$

Here the  $\ell$ -th components of  $\mathbf{a}_j^i$ 's are defined inductively by

$$\lambda_i(\mathbf{a}_j^i)_\ell = \left( F^{(\lambda_j)} \mathbf{a}_j^i - \sum_{\ell=j}^{i-1} T_{\mathbf{a}_j^\ell} \mathbf{a}_\ell^i \right)_\ell.$$

We put

$$H_i := \frac{1}{\lambda_{i+1}} \left( \frac{D_i(X_0, \dots, X_{i-1}) D_i(Y_0, \dots, Y_{i-1})}{D_i((X_0, \dots, X_{i-1}) + (Y_0, \dots, Y_{i-1}))} - 1 \right).$$

Furthermore we define a formal power series by

$$\begin{aligned} & F_p \left( \begin{pmatrix} \mathbf{b}_1^i \\ \mathbf{b}_2^i \\ \vdots \\ \mathbf{b}_i^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1}, Y_0, Y_1, \dots, Y_{i-1} \right) \\ &:= \prod_{j=1}^i F_p(\mathbf{b}_j^i, \lambda_j; \frac{X_{j-1}}{D_{j-1}(X_0, \dots, X_{j-2})}, \frac{Y_{j-1}}{D_{j-1}(Y_0, \dots, Y_{j-2})}) \times \\ & \quad \prod_{k=2}^i F_p(\mathbf{b}_k^i, \lambda_k; H_{k-1}, \frac{X_{k-1}}{D_{k-1}(X_0, \dots, X_{k-2})} + \frac{Y_{k-1}}{D_{k-1}(Y_0, \dots, Y_{k-2})}) \times \\ & \quad G_p(-\mathbf{b}_k^i, \lambda_k; \lambda_k H_{k-1} + 1). \end{aligned}$$

Then we can show the following theorem inductively.

**Theorem 4.4.** *The homomorphisms*

$$\begin{aligned} \xi_0^i : \text{Ker}(\widehat{W}(A)^i \xrightarrow{U_i} \widehat{W}(A)^i) &\rightarrow \text{Hom}(\mathcal{W}_i, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^i \end{pmatrix} &\mapsto E_p \left( \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \beta^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1} \right) \end{aligned}$$

and

$$\begin{aligned} \xi_1^i : \text{Coker}(\widehat{W}(A)^i \xrightarrow{U_i} \widehat{W}(A)^i) &\rightarrow H_0^2(\mathcal{W}_i, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^i \end{pmatrix} &\mapsto F_p \left( \begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1}, Y_0, Y_1, \dots, Y_{i-1} \right) \end{aligned}$$

are bijective, for each  $i$ .

## REFERENCES

1. DEMAZURE, M. and GABRIEL, P., *Groupes algébriques, Tome 1*, Masson-North-Holland, 1970
2. HAZEWINKEL, M., *Formal groups and applications*, Academic Press, 1978
3. ILLUSIE, L., *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. de l'Ec. Norm. Sup. 4<sup>e</sup> série 12, 501–661(1979)
4. GREEN, B. and MATIGNON, M., *Liftings of Galois covers of smooth curves*, to appear in Compositio Math.
5. LAZARD, M., *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France, 83, 251–274(1955)
6. SERRE, J.-P., *Groupes algébriques et corps de classes*, Hermann, Paris, 1959
7. SEKIGUCHI, T. and OORT, F. and SUWA, N., *On the deformation of Artin-Schreier to Kummer*, Ann. Scient. Éc. Norm. Sup., 4<sup>e</sup> série 22, 345–375(1989)
8. SEKIGUCHI, T. and SUWA, N., *A case of extensions of group schemes over a discrete valuation ring*, Tsukuba J. Math., 14, No. 2, 459 – 487(1990)
9. SEKIGUCHI, T. and SUWA, N., *Some cases of extensions of group schemes over a discrete valuation ring I*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., 38, 1 – 45(1991)
10. SEKIGUCHI, T. and SUWA, N., *Some cases of extensions of group schemes over a discrete valuation ring II*, Bull. Facul. Sci. & Eng., Chuo University, 32, 17 –36(1989)
11. SEKIGUCHI, T. and SUWA, N., *Théorie de Kummer-Artin-Schreier*, C. R. Acad. Sci. Paris, 312, Série I, 417 –420(1991)
12. SEKIGUCHI, T. and SUWA, N., *Théorie de Kummer-Artin-Schreier-Witt*, C. R. Acad. Sci. Paris, 319, Série I, 105–110(1994)
13. SEKIGUCHI, T. and SUWA, N., *A note on extensions of algebraic and formal groups I, II*, Math. Z., 206, 567–575(1991), 217, 447–457(1994)
14. SEKIGUCHI, T. and SUWA, N., *A note on extensions of algebraic and formal groups III*, Tôhoku Math. J. 49(1997), 241–257
15. SEKIGUCHI, T. and SUWA, N., *On the unified Kummer-Artin-Schreier-Witt theory*, Chuo Math Preprint Series 41, 1994
16. SEKIGUCHI, T. and SUWA, N., *A note on extensions of algebraic and formal groups IV*, preprint
17. WATERHOUSE, W. AND WEISFEILER, B., *One-dimensional affine group schemes*, J. of Alg., 66, 550–568(1980)

(Tsutomu Sekiguchi) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, CHUO UNIVERSITY 1-13-27 KASUGA BUNKYO-KU TOKYO 112-8551, JAPAN

*E-mail address:* sekiguti@math.chuo-u.ac.jp

(Noriyuki Suwa) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, CHUO UNIVERSITY 1-13-27 KASUGA BUNKYO-KU TOKYO 112-8551, JAPAN

*E-mail address:* suwa@math.chuo-u.ac.jp